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Stability of plane Poiseuille flow with temperature dependent viscosity

P. SCHÄFER and H. HERWIG
Institut für Thermo- und Fluiddynamik, Ruhr-Universität, 4630 Bochum, F.R.G.

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Abstract—Classical linear stability theory is extended to include the effect of temperature dependent viscosity. This effect is studied asymptotically by using a Taylor series expansion of viscosity with respect to temperature. In its general form the asymptotic solution holds for all Newtonian fluids for which the temperature dependence of viscosity is the dominating variable property effect. A shooting technique with Gram–Schmidt orthonormalization for the zero-order equation (classical Orr–Sommerfeld problem) and a multiple shooting method for all other equations are applied to solve the stiff differential equations.

1. INTRODUCTION

AMONG the studies that have investigated the stability of laminar boundary layer flows, only a few have taken into account the effect of variable properties, even though non-constant properties can have a strong effect on the critical Reynolds number. For example, Wazzan *et al.* [1] investigated the boundary layer stability of water under non-isothermal conditions. They found that the critical Reynolds number for a heated flat plate boundary layer in water varies between 520 and nearly 16 000. Potter and Graber [2] in their study for liquids found a reduction of the critical Reynolds number by 50% for a 78°C temperature difference between the walls. Other studies of forced convection stability which take into account variable property effects in a more or less systematic way are those by Hauptmann [3], Lee *et al.* [4] and Asfar *et al.* [5].

The general method to account for small variable property effects, which will be applied to Poiseuille flow with temperature dependent viscosity, is outlined in Herwig and Schäfer [6]. The basic approach starts from a Taylor series expansion of the properties under consideration. Next, a regular perturbation procedure is applied to the basic equations of stability with the constant property case representing leading order behaviour.

2. GOVERNING EQUATIONS

The flow under consideration is a laminar plane Poiseuille flow which is heated for $x^* > 0$ with a constant wall heat flux which is equal on both walls ($q^*_{wu} = -q^*_{wl}$), see Fig. 1. Due to the temperature dependence of viscosity the velocity profile is changed by the heat transfer. Downstream of an adjustment zone the flow will reach a new fully developed state which is described in paragraph 3.1 below. In Fig. 1 typical velocity and temperature profiles are shown in this section. The flow for $x^* > 0$ is fully developed as soon as the difference between the wall and the bulk temperature, i.e. $(T^*_w - T^*_B)$, is independent of x^*, see the lower part of Fig. 1. Stability considerations of this study refer to this part of the flow field.

2.1. *Mean flow equations*

The basic equations for the mean flow are the so-called slender channel equations for variable viscosity, see for example Van Dyke [7] and Gersten and Herwig [8]. Nondimensionalised according to Table 1 they read ($-\ \hat{=}$ mean flow quantity)

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\mathrm{d}\bar{p}}{\mathrm{d}x} + \frac{\partial}{\partial y}\left(\bar{\mu}\frac{\partial \bar{u}}{\partial y}\right) \tag{2}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} = \frac{1}{Pr_R}\frac{\partial^2 \bar{T}}{\partial y^2} \tag{3}$$

with the associated boundary conditions

$$y = 1 : \bar{u} = \bar{v} = 0, \quad \frac{\partial \bar{T}}{\partial y} = 1 \tag{4}$$

$$y = -1 : \bar{u} = \bar{v} = 0, \quad \frac{\partial \bar{T}}{\partial y} = -1. \tag{5}$$

The reference state R should be at $x^* = 0$ when the whole region $x^* > 0$ is under consideration. But, since we only refer to the fully developed region, we may (arbitrarily) choose some position x^*_R in this part of the flow field with $T^*_B(x^*_R)$ as reference temperature. This can be done because the x^*-dependence of $T^*(x^*, y^*)$ is of the special kind described by equation (36) below.

With the reference velocity $U^*_R = \dot{m}^*/(\rho^* B^* 2H^*)$ (average velocity in the channel) the Reynolds number (with $\mu^*_R = \mu^*(T^*_B)$) is

NOMENCLATURE

c_i	amplification rate
c_r	phase velocity
c_p	specific heat at constant pressure
H	half channel height
k	thermal conductivity
K_μ	nondimensional viscosity gradient, equation (16)
$\dot{m}$	mass flux
p	pressure
Pr	Prandtl number $\mu^* c_p^*/k^*$
q_w	wall heat flux
Re	Reynolds number, equation (6)
T	temperature
$\hat{T}$	temperature amplitude function
u	streamwise velocity
$\hat{u}$	amplitude function of the streamwise velocity
U_R	reference velocity
v	velocity normal to the wall
$\hat{v}$	amplitude function of the velocity normal to the wall
x, y	Cartesian coordinates.

Greek symbols

α	wave length parameter, equation (7)
ε	perturbation parameter, equation (16)
μ	viscosity
$\hat{\mu}$	amplitude function of the viscosity
φ	stream function
ρ	density.

Superscripts

*	dimensional quantity
—	mean value
′	fluctuating quantity
′	derivative with respect to y
^	complex quantity.

Subscripts

B	bulk
c	critical
i	imaginary part
l	lower
r	real part
R	reference condition
u	upper
w	wall
0	zero order
1	first order.

$$Re_R = \frac{\rho^* U_R^* H^*}{\mu_R^*}. \tag{6}$$

2.2. *Linear stability equations*

For constant properties, the fundamental differential equation of the method of small disturbances is the Orr–Sommerfeld equation (from now on referred to as the OS equation), see for example Schlichting [9]. Squire [10] has shown that it is sufficient to consider two-dimensional disturbances. For variable properties, the same result was found by Yih [11] by extending Squire's method.

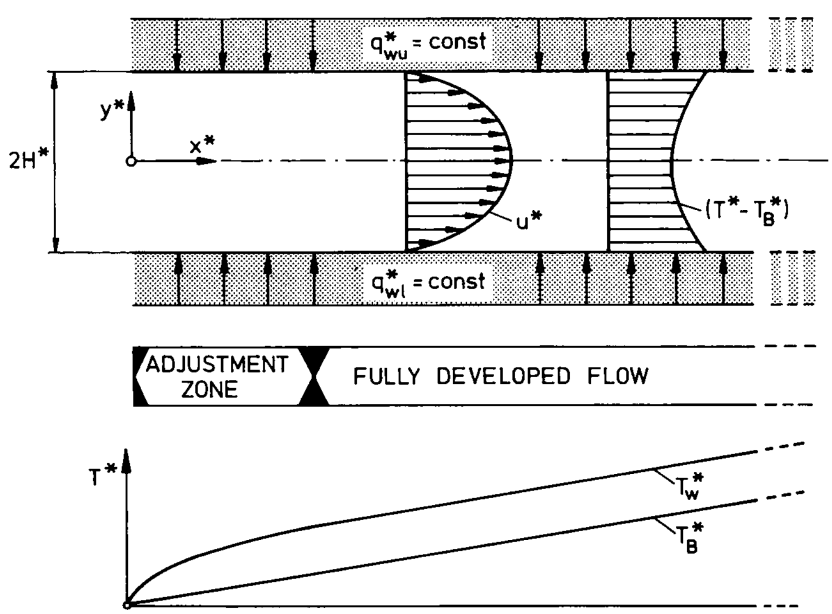

FIG. 1. Velocity and temperature profiles in the fully developed region of a uniformly heated channel.

Table 1. Nondimensionalization in equations (1)–(5). (All dimensional quantities are starred)

x	y	u	v	p	T	μ
$\dfrac{x^*-x^*_R}{H^*\,Re_R}$	$\dfrac{y^*}{H^*}$	$\dfrac{u^*}{U^*_R}$	$\dfrac{v^*\,Re_R}{U^*_R}$	$\dfrac{p^*-p^*_R}{\rho^* U^{*2}_R}$	$\dfrac{T^*-T^*_R}{q^*_{wl}H^*/k^*}$	$\dfrac{\mu^*}{\mu^*_R}$

For the present analysis we need an extended version of the OS equation which holds when viscosity is temperature dependent. Due to this temperature dependence the modified OS equation must be supplemented by the thermal energy equation for the disturbances.

In the method of small disturbances all quantities are decomposed into a mean value, $\bar{a}^*$, and a superimposed disturbance a'^*. Here a^* represents the velocity components u^* and v^* (two-dimensional flow), the pressure p^*, the temperature T^* and the viscosity μ^*. The common assumption (e.g. Schlichting [9]) is that an arbitrary two-dimensional disturbance can be expanded in a Fourier series, thus a single oscillation of the disturbance is assumed to be of the form (temporal stability)

$$\hat{a}'^*(x^*,y^*,t^*) = \hat{a}^*(y^*)\exp[\mathrm{i}\,\alpha^*(x^*-\hat{c}^*t^*)]. \quad (7)$$

In equation (7) α^* is real with $2\pi/\alpha^*$ being the wavelength of the single oscillation. The quantity $\hat{c}^*$ is complex,

$$\hat{c}^* = c^*_r + \mathrm{i}\,c^*_i. \quad (8)$$

Here c^*_r denotes the phase velocity whereas c^*_i determines the degree of amplification ($c^*_i > 0$) or damping ($c^*_i < 0$). From now on all complex quantities are marked by the symbol ˆ.

From the Navier–Stokes equations and the thermal energy equation (both for temperature dependent viscosity), together with the continuity equation the differential equations for the dimensionless amplitude functions $\hat{u}(y)$, $\hat{v}(y)$ and $\hat{T}(y)$ can be deduced. For details of this procedure see the related study by Herwig and Schäfer [6].

We introduce the stream function $\hat{\varphi}$ by

$$\hat{u} = \hat{\varphi}', \quad \hat{v} = -\mathrm{i}\,\alpha\hat{\varphi}. \quad (9)$$

Here and from now on the symbol ′ denotes the derivative with respect to y, i.e. for example $\hat{\varphi}' = \partial\hat{\varphi}/\partial y$. The two differential equations for $\hat{\varphi}(y)$ and $\hat{T}(y)$ are, respectively,

$$\begin{aligned}(\bar{u}-\hat{c})(\hat{\varphi}''-\alpha^2\hat{\varphi})-\bar{u}''\hat{\varphi} &= \frac{-\mathrm{i}}{\alpha\,Re_R}\{\bar{\mu}(\hat{\varphi}''''-2\alpha^2\hat{\varphi}''+\alpha^4\hat{\varphi}) \\ &\quad +2\bar{\mu}'(\hat{\varphi}'''-\alpha^2\hat{\varphi}')+\bar{\mu}''(\hat{\varphi}''+\alpha^2\hat{\varphi}) \\ &\quad +\hat{\mu}(\bar{u}'''+\alpha^2\bar{u}')+2\hat{\mu}'\bar{u}''+\hat{\mu}''\bar{u}'\} \\ &\quad +\frac{1}{Re_R^2}\frac{\partial\bar{\mu}}{\partial x}(\hat{\varphi}''-\alpha^2\hat{\varphi}) \quad (10)\end{aligned}$$

$$(\bar{u}-\hat{c})\hat{T}-\bar{T}'\hat{\varphi} = \frac{-\mathrm{i}}{\alpha\,Re_R\,Pr_R}(\hat{T}''-\alpha^2\hat{T}) + \frac{\mathrm{i}}{\alpha\,Re_R}\frac{\partial\bar{T}}{\partial x}\hat{\varphi}'. \quad (11)$$

In equations (10) and (11) the amplitude functions $\hat{\varphi}$, $\hat{T}$ and $\hat{\mu}$ are nondimensionalised like the corresponding mean flow quantities, the parameters α and $\hat{c}$ are $\alpha = \alpha^*H^*$ and $\hat{c} = \hat{c}^*/U^*_R$, respectively. For $\bar{\mu} = 1$ and $\hat{\mu} = 0$, equation (10) is reduced to the well-known OS equation (constant properties).

In equations (10) and (11) quadratic terms are neglected (linear stability theory). In this case (q^*_w = const.) we need no parallel flow assumption because the only x-dependences in the governing stability equations are those of $T_B(x)$ and $\bar{\mu}(x)$ which are linear functions, so that $\partial T/\partial x$ and $\partial\bar{\mu}/\partial x$ are constant.

The boundary conditions for $\hat{\varphi}$ and $\hat{T}$ are

$$y = 1: \hat{\varphi} = \hat{\varphi}' = \hat{T} = 0 \quad (12)$$

$$y = -1: \hat{\varphi} = \hat{\varphi}' = \hat{T} = 0. \quad (13)$$

3. ASYMPTOTIC APPROACH

The influence of temperature dependent viscosity will be accounted for by a regular perturbation procedure based on a Taylor series expansion of $\mu^*(T^*)$. This has been proved to be successful for laminar (mean) flow, see for example Herwig [12] for pipe flow with variable properties. It will be extended in this study to the stability problem. The Taylor series expansion of μ^* at the reference temperature T^*_R is

$$\mu^* = \mu^*_R + \left.\frac{\mathrm{d}\mu^*}{\mathrm{d}T^*}\right|_R (T^*-T^*_R) + \cdots. \quad (14)$$

Equation (14) can be nondimensionalised and rewritten as follows

$$\mu = \frac{\mu^*}{\mu^*_R} = 1+\varepsilon K_\mu T + \mathrm{O}(\varepsilon^2) \quad (15)$$

with

$$K_\mu = \left[\frac{\mathrm{d}\mu^*}{\mathrm{d}T^*}\frac{T^*}{\mu^*}\right]_R; \quad \varepsilon = \frac{q^*_{wl}H^*}{\lambda^* T^*_R}. \quad (16)$$

The nondimensional temperature T according to Table 1 is $T = (T^*-T^*_R)/(q^*_{wl}H^*/\lambda^*)$. In equation (15) ε is assumed to be small, i.e. we assume small heat transfer rates. The nondimensional heat flux ε will be the perturbation parameter for the subsequent asymptotic analysis. Truncating the Taylor series after the linear term, as we do in equation (15), results in a linear perturbation theory with respect to ε. Extension to higher orders ($\varepsilon^2, \varepsilon^3, \ldots$) is straightforward but not in the scope of this study.

The parameter K_μ is the nondimensional first derivative of μ^* with respect to T^*. It is a property of the fluid (for example: water at $T^*_R = 293$ K, $K_\mu = -7.13$).

Due to the decomposition $\mu = \bar{\mu} + \hat{\mu} \exp[i\alpha(x-\hat{c}t)]$ equation (15) results in

$$\bar{\mu} = 1 + \varepsilon K_\mu \bar{T} + O(\varepsilon^2) \tag{17}$$

$$\hat{\mu} = \varepsilon K_\mu \hat{T} + O(\varepsilon^2). \tag{18}$$

The mean flow field is affected by variable viscosity effects through $\bar{\mu}$ whereas the stability equations (10) and (11) are affected by the mean as well as by the disturbance part of the viscosity.

3.1. *Mean flow solutions*

Equation (17) suggests the following expansions of the mean flow quantities:

$$\begin{aligned}
\bar{u} &= \bar{u}_0 + \varepsilon K_\mu \bar{u}_1 + O(\varepsilon^2) \\
\bar{v} &= \bar{v}_0 + \varepsilon K_\mu \bar{v}_1 + O(\varepsilon^2) \\
\bar{p} &= \bar{p}_0 + \varepsilon K_\mu \bar{p}_1 + O(\varepsilon^2) \\
\bar{T} &= \bar{T}_0 + \varepsilon K_\mu \bar{T}_1 + O(\varepsilon^2).
\end{aligned} \tag{19}$$

Inserting these expansions into equations (1)–(3) and collecting terms of equal magnitude with respect to ε (i.e. O(1), O(ε)) gives two sets of equations, one for the zero order quantities $\bar{u}_0$, $\bar{v}_0$, ... and one for $\bar{u}_1$, $\bar{v}_1$, ...

These equations can be solved analytically (see Herwig [12] for the related study of pipe flow). The solutions are:

$$\bar{u}_0 = \tfrac{3}{2}(1-y^2); \quad \bar{v}_0 = 0 \tag{20}$$

$$\bar{p}_0 = -3x \tag{21}$$

$$\bar{T}_0 = \frac{3}{2}\left(-\frac{1}{12}y^4 + \frac{1}{2}y^2 - \frac{13}{140}\right) \tag{22}$$

and

$$\bar{u}_1 = \frac{3}{2}\left(-\frac{1}{24}y^6 + \frac{3}{8}y^4 - \frac{111}{280}y^2 + \frac{53}{840}\right); \quad \bar{v}_1 = 0 \tag{23}$$

$$\bar{p}_1 = -\frac{27}{35}x - \frac{3}{2\,Pr_R}x^2 \tag{24}$$

$$\bar{T}_1 = \frac{1}{2}\left(-\frac{1}{448}y^8 + \frac{3}{80}y^6 - \frac{111}{1120}y^4 + \frac{53}{560}y^2 + \frac{16\,817}{2\,587\,200}\right). \tag{25}$$

Equations (20)–(22) are the well-known constant property results. Equations (23)–(25) are the variable viscosity influence functions with respect to (small) heat transfer rates. Together with the constant property results, according to equation (19), they describe the mean flow in the fully developed heat transfer region of Fig. 1.

3.2. *Linear stability solutions*

The stability equations (10) and (11), from which the amplitude functions $\hat{\varphi}$ and $\hat{T}$ can be determined, are now subject to a perturbation procedure similar to that of the mean flow. Therefore, also $\hat{\varphi}$, $\hat{T}$ and $\hat{c}$ are expanded:

$$\begin{aligned}
\hat{\varphi} &= \hat{\varphi}_0 + \varepsilon K_\mu \hat{\varphi}_1 + O(\varepsilon^2) \\
\hat{T} &= \hat{T}_0 + \varepsilon K_\mu \hat{T}_1 + O(\varepsilon^2) \\
\hat{c} &= \hat{c}_0 + \varepsilon K_\mu \hat{c}_1 + O(\varepsilon^2).
\end{aligned} \tag{26}$$

A crucial step in the theory is the expansion of the parameter $\hat{c}$ in the same way as the expansion of $\hat{\varphi}$ and $\hat{T}$. This leads to the specific form of the first order equation (30), below, from which $\hat{c}_1$ can be determined.

The viscosity amplitude function $\hat{\mu}$ according to equation (18) now is

$$\hat{\mu} = \varepsilon K_\mu \hat{T}_0 + O(\varepsilon^2). \tag{27}$$

Inserting all the expansions into (10) and (11), and collecting terms of equal magnitude with respect to εK_μ, leads to the following set of stability equations:

Zero order:

$$(\bar{u}_0 - \hat{c}_0)(\hat{\varphi}_0'' - \alpha^2\hat{\varphi}_0) - \bar{u}_0''\hat{\varphi}_0 = -\frac{1}{\alpha\, Re_R}(\hat{\varphi}_0'''' - 2\alpha^2\hat{\varphi}_0'' + \alpha^4\hat{\varphi}_0) \tag{28}$$

$$(\bar{u}_0 - \hat{c}_0)\hat{T}_0 + \frac{i}{\alpha\, Re_R\, Pr_R}(\hat{T}_0'' - \alpha^2\hat{T}_0) = \bar{T}_0'\hat{\varphi}_0 + \frac{i}{\alpha\, Re_R\, Pr_R}\hat{\varphi}_0'. \tag{29}$$

First order:

$$\begin{aligned}
&(\bar{u}_0 - \hat{c}_0)(\hat{\varphi}_1'' - \alpha^2\hat{\varphi}_1) - \bar{u}_0''\hat{\varphi}_1 \\
&\quad + \frac{i}{\alpha\, Re_R}(\hat{\varphi}_1'''' - 2\alpha^2\hat{\varphi}_1'' + \alpha^4\hat{\varphi}_1) \\
&= -(\bar{u}_1 - \hat{c}_1)(\hat{\varphi}_0'' - \alpha^2\hat{\varphi}_0) + \bar{u}_1''\hat{\varphi}_0 \\
&\quad - \frac{i}{\alpha\, Re_R}[\bar{T}_0(\hat{\varphi}_0'''' - 2\alpha^2\hat{\varphi}_0'' + \alpha^4\hat{\varphi}_0) \\
&\quad + 2\bar{T}_0'(\hat{\varphi}_0''' - \alpha^2\hat{\varphi}_0') + \bar{T}_0''(\hat{\varphi}_0'' + \alpha^2\hat{\varphi}_0) \\
&\quad + \hat{T}_0(\bar{u}_0''' + \alpha^2\bar{u}_0') + 2\bar{u}_0''\hat{T}_0' + \bar{u}_0'\hat{T}_0''] \\
&\quad + \frac{1}{Re_R^2\, Pr_R}(\hat{\varphi}_0'' - \alpha^2\hat{\varphi}_0)
\end{aligned} \tag{30}$$

with the associated boundary conditions

$$y = +1: \hat{\varphi}_0 = \hat{\varphi}_0' = \hat{T}_0 = \hat{\varphi}_1 = \hat{\varphi}_1' = 0 \tag{31}$$

$$y = -1: \hat{\varphi}_0 = \hat{\varphi}_0' = \hat{T}_0 = \hat{\varphi}_1 = \hat{\varphi}_1' = 0. \tag{32}$$

The terms $\partial\bar{\mu}/\partial x$ and $\partial\bar{T}/\partial x$ in equations (10) and (11) are $\partial\bar{\mu}/\partial x = \varepsilon K_\mu/Pr_R$ and $\partial\bar{T}/\partial x = 1/Pr_R$, respectively.

Equation (28) is the classical OS equation (constant properties). It is the zero order equation of the asymptotic expansion with respect to ε. It is well-known that the OS equation describes an eigenvalue problem which owing to its stiffness is difficult to solve numerically (see e.g. Mack [13]). With these difficulties in mind, the numerical procedure for the zero and first order equations will be discussed next.

3.2.1. *Zero order momentum (classical OS equation)*. Due to the stiffness of equation (28), the integration is performed by applying the so-called *Gram–Schmidt orthonormalization.* For details of this procedure, see for example Mack [14]. As a consequence of this orthonormalization $\hat{\varphi}_0(y)$ is given as a piecewise steady function in subregions of $-1 \leqslant y \leqslant 1$. But, since $\hat{\varphi}_0(y)$ is needed in equation (29), a continuous function is recovered from this by a patching procedure described in Herwig and Schäfer [6]. As an example for this continuous function the derivative of $\hat{\varphi}_0$ (i.e. $d\varphi_{0r}/dy$ and $d\varphi_{0i}/dy$) is shown in Fig. 2 for the critical Reynolds number Re_{c0}. Since $\hat{\varphi}_0$ can be determined from equation (28) only up to a (complex) constant, $\hat{\varphi}_0$ can be arbitrarily normalised.

The critical Reynolds number from our calculations is $Re_{c0} = 3848.1$ which agrees very well with the result of Orszag [15], for example, who also found $Re_{c0} = 3848.1$.

In Fig. 2 the y-position of the critical layer (i.e. the position where $\bar{u}_0 = c_{0r}$) is marked by an arrow. At this position, the stability equations become singular for $1/Re_R = 0$, see equation (28). For large but finite Reynolds numbers substantial changes may occur in the vicinity of this layer.

3.2.2. *Zero order temperature.* The thermal energy stability equation (29) is solved to determine the amplitude function $\hat{T}_0(y)$. Equation (29) is a nonhomogeneous linear second order differential equation with homogeneous boundary conditions. It is also a stiff differential equation like (28), and was solved by the so-called *multiple shooting method* (see Stoer and Bulirsch [16]). In this method, the whole solution domain is cast into subregions. Then a first step integration is performed starting from assumed boundary conditions in each subregion (taking into account the boundary conditions on both walls). In subsequent steps, the discontinuities at the boundaries of the subregions are removed so that a continuous function $\hat{T}_0$ results.

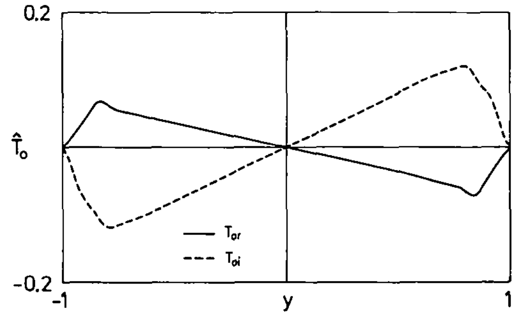

FIG. 3. Zero order temperature amplitude function $\hat{T}_0$ for $q_w^* = \text{const.}$ $Re_R = 3848.1$; $\alpha = 1.02$; $c_{0r} = 0.396$; $c_{0i} = 0.0$ $Pr_R = 0.7$.

In Fig. 3, the amplitude function $\hat{T}_0(y)$ is shown for a specific set of parameters (Pr_R, Re_R, c_{0i}).

3.2.3. *First order momentum.* Equation (30) is a nonhomogeneous differential equation of the general form

$$L[\hat{\varphi}_1, \hat{c}_0] = f(\hat{\varphi}_0, \hat{T}_0, \hat{c}_1) \tag{33}$$

with the OS differential operator L given by

$$L[\hat{\varphi}, \hat{c}_0] = (\bar{u}_0 - \hat{c}_0)(\hat{\varphi}'' - \alpha^2\hat{\varphi}) - \bar{u}_0''\hat{\varphi} + \frac{i}{\alpha\, Re_R}(\hat{\varphi}'''' - 2\alpha^2\hat{\varphi}'' + \alpha^4\hat{\varphi}). \tag{34}$$

Specific values of $\hat{c}_1$ must be found for which (33) has a solution. The term 'eigenvalue' should be used only in connection with homogeneous equations. Therefore $\hat{c}_1$ will be called 'first order parameter' from now on.

The corresponding solution will be denoted by $\hat{\varphi}_{1p}$ (p for particular solution). The general solution of equation (33) then is

$$\hat{\varphi}_1 = \hat{\varphi}_{1p} + \hat{C}\hat{\varphi}_0, \tag{35}$$

since $\hat{\varphi}_0$ satisfies $L[\hat{\varphi}, \hat{c}_0] = 0$. Due to the undetermined complex constant $\hat{C}$ in (35) integration can start from the lower wall fixing $\hat{\varphi}_1''(-1)$ arbitrarily, for example. Integration was again performed by the multiple shooting method, using $\hat{c}_1$ as the shooting parameter. In Fig. 4 the first derivative of the amplitude function $\hat{\varphi}_1(y)$ is shown for the same parameters as in Fig. 3. For normalization we have set $\varphi_{1r}'' = \varphi_{1i}'' = 1$. The constant $\hat{c}_1$ for this case is $0.0374 + i \cdot 0.0175$.

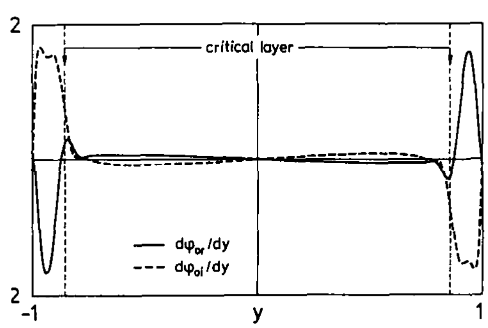

FIG. 2. First derivatives of the amplitude function $\hat{\varphi}_0$. $Re_R = 3848.1$; $\alpha = 1.02$; $c_{0r} = 0.396$; $c_{0i} = 0.0$. Normalised so that max $|\varphi_{0r}'| = $ max $|\varphi_{0i}'| = 1$.

3.3. *The critical Reynolds number*

The influence of temperature dependent viscosity on the critical Reynolds number can be determined

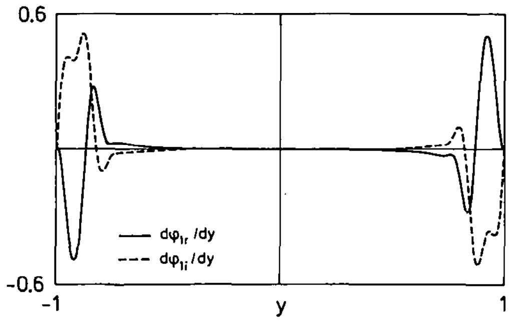

FIG. 4. First derivative of the amplitude function $\hat{\varphi}_1$. Re_R, α, $\hat{c}_0$ and Pr_R as in Fig. 3. Normalised, so that $\varphi''_{1r}(0) = \varphi''_{1i}(0) = 1$.

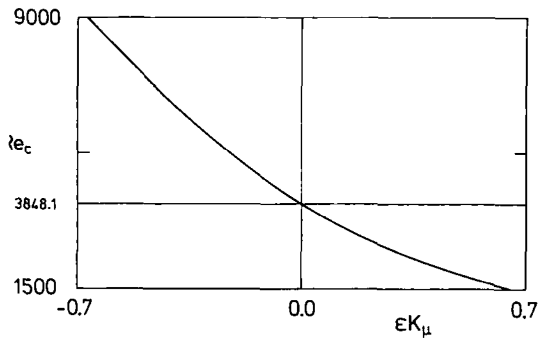

FIG. 6. Critical Reynolds number for non-isothermal Poiseuille flow with temperature dependent viscosity; $q^*_w = \text{const.}$; $Pr = 0.7$ (very insensitive with respect to this parameter, see Section 3.3). $Re_c = \rho^* U^*_R H^*/\mu^*_R$ with $\mu^*_R = \mu^*(T^*_B(x^*_R))$.

from Figs. 5(a)–(c). In these figures the eigenvalue c_{0i} and the first order parameter c_{1i} are given as functions of α for three different Reynolds numbers. Flow instabilities occur, whenever $c_i = c_{0i} + \varepsilon K_\mu c_{1i}$ exceeds zero. At the critical Reynolds number $c_i = 0$ holds just for one particular value of α. For constant properties ($\varepsilon = 0$, i.e. $c_i = c_{0i}$) this occurs for $Re = 3848.1$, as can be seen in Fig. 5(b). For variable properties, the critical Reynolds number Re_c is reached, when $c_i = c_{0i} + \varepsilon K_\mu c_{1i} = 0$. From Fig. 5(a) we find that $Re_c = 1000$ for $\varepsilon K_\mu = 1.11$ since then c_i is zero for just one α. The corresponding numbers in Fig. 5(c) are: $Re_c = 8000$ for $\varepsilon K_\mu = -0.71$.

Based on a large number of curves for $c_{0i}(\alpha)$ and $c_{1i}(\alpha)$ like those in Fig. 5, we can find the critical Reynolds number Re_c as a function of the perturbation parameter εK_μ (here K_μ is an $O(1)$ constant with respect to the asymptotic expansions). In Fig. 6, Re_c is shown as a function of εK_μ. From equations (29) and (30) one expects a Prandtl number dependence of these curves. But, it turns out that this effect is so small that in Fig. 6 the two curves for $Pr = 0.7$ and 7.0, for example, coincide. The Prandtl number only influences the temperature fluctuations (which have little influence on the stability behaviour, see Section 4.1 below).

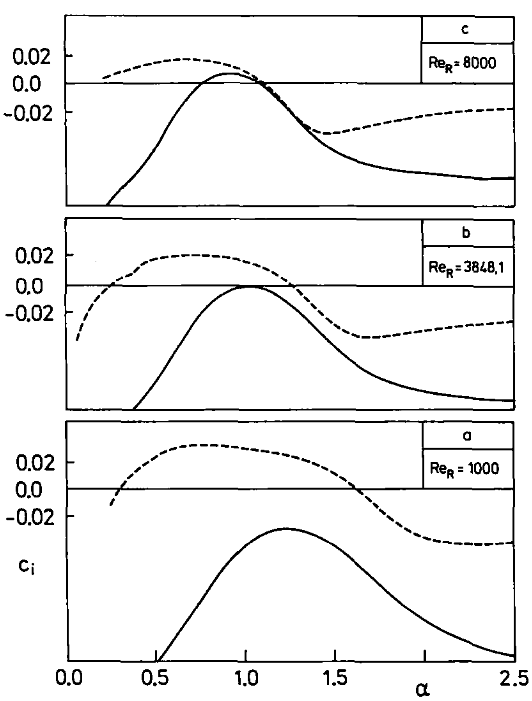

FIG. 5. Eigenvalues c_{0i} and first order parameters c_{1i} for three different Reynolds numbers, $Pr = 0.7$, $q^*_w = \text{const.}$ ——— Eigenvalue c_{0i}. --- First order parameter c_{1i}.

From this figure we can conclude that the flow is stabilised when $\varepsilon K_\mu < 0$, since then the critical Reynolds number lies above that for $\varepsilon K_\mu = 0$ (classical OS problem).

The combination εK_μ is negative:

(a) for fluids with $K_\mu < 0$ that are heated ($\varepsilon > 0$) (like water with $K_\mu = -7.13$ at 293 K, 1 bar);
(b) for fluids with $K_\mu > 0$ that are cooled ($\varepsilon < 0$) (like air with $K_\mu = 0.73$ at 293 K, 1 bar).

From Fig. 6 it can also be concluded that the stabilising/destabilising effect for a certain amount of heating (fixed $\varepsilon > 0$) for water is much stronger than for air, since $|K_\mu|$ is nearly ten times larger for water.

For a physical interpretation of these effects, one should keep in mind the temperature distribution which is sketched in Fig. 1. In the fully developed region the temperature can be split into two terms, one being x- and the other being y-dependent, i.e.

$$T^*(x^*, y^*) = T^*_B(x^*) + T^*_y(y^*),$$
$$T^*_y(y^*) = T^*(x^*, y^*) - T^*_B(x^*) \qquad (36)$$

or, nondimensionalised with a yet undetermined reference temperature T^*_R,

$$\underbrace{\frac{T^* - T^*_R}{q^*_{w1} H^*/\lambda^*}}_{T(x,y)} = \underbrace{\frac{T^*_B - T^*_R}{q^*_{w1} H^*/\lambda^*}}_{T_B(x)} + \underbrace{\frac{T^* - T^*_B}{q^*_{w1} H^*/\lambda^*}}_{T_y(y)} . \qquad (37)$$

Energy conservation between two cross sections x^*_R and $x^* > x^*_R$ leads to $T_B(x) = x/Pr_R$ when

$T_R^* = T_B^*(x_B^*)$ and $x = (x^* - x_R^*)/(H^* Re_R)$ according to Table 1, so that finally

$$T = \frac{x}{Pr_R} + T_y. \qquad (38)$$

At the reference position x_R^*, i.e. at $x = 0$, the temperature profile T is $T = T_y$. Since the whole analysis of our study holds at this reference position, we simply used T for the nondimensional temperature instead of T_y.

From these considerations we can conclude that the viscosity effect on Re_c, shown in Fig. 6, is only caused by the y-distribution of temperature and viscosity! The variation in x, which actually only means an increase of temperature level with increasing x, is accounted for in the zero order already (by changing Re_R and Pr_R). This on the other hand means that a term 'constant property case' for the zero order (classical OS problem) would be misleading. We suggest calling it the 'quasi constant property case', see also Herwig [12]. It means that the zero order results are applied locally, i.e. with the local bulk temperature as reference temperature. As far as the zero order critical Reynolds number Re_{c0} is concerned this means that the viscosity μ_R^* in the definition of Re_R according to (6) is $\mu_R^* = \mu^*(T_B^*(x_R^*))$.

4. DISCUSSION

4.1. *Influence of temperature fluctuations*

The common treatment of temperature fluctuations is just to neglect them, see for example Wazzan *et al.* [1]. This, however, cannot be justified from an asymptotic point of view. It is correct only in the limit $Re_R^{-1} = 0$, see equation (10). However, numerically the influence is extremely small in the case considered in this study.

4.2. *Comparison with non-asymptotic results*

Our final asymptotic results can be specified for a certain fluid by specifying K_μ and for a certain temperature difference by specifying ε. Thus they can be compared to the results of other studies which assume a certain viscosity from the beginning and calculate specific results for a finite number of different heat transfer cases.

Since a study by Potter and Graber [2] is closely related to our flow and heat transfer situation, we recalculated their case with our asymptotic method. Basically it is a change in thermal boundary conditions that was necessary compared to what we have done so far. Instead of $q_w^* =$ const., they chose a constant (though different) wall temperature on both sides of the wall. The perturbation parameter for this case is

$$\varepsilon = \frac{T_{wu}^* - T_{wl}^*}{T_{wl}^*} \qquad (39)$$

with T_{wu}^* and T_{wl}^* as temperatures of the upper and lower wall, respectively. The reference temperature now is $T_R^* = T_{wl}^*$.

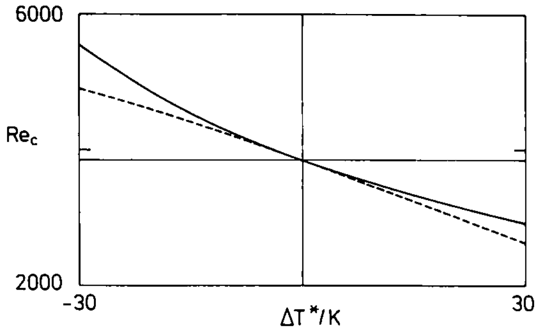

FIG. 7. Critical Reynolds number for a Poiseuille flow with $T_{wu}^* =$ const. and $T_{wl}^* =$ const. Exponential viscosity law. ——— Non-asymptotic (Potter and Graber [2]). --- Asymptotic (this study).

The study of Potter and Graber holds for water, for which they assumed the exponential viscosity law

$$\mu^* = c_2 \mu_0^* \exp[c_1/T^*] \qquad (40)$$

with two constants c_1 and c_2 and the viscosity at the cold wall, μ_0^*. From this equation we immediately get K_μ according to equation (16),

$$K_\mu = -c_1/T_{wl}^*. \qquad (41)$$

In Fig. 7 the critical Reynolds number is shown as a function of the temperature difference $\Delta T^* = T_{wu}^* - T_{wl}^*$. The increasing deviations for an increasing temperature difference are higher order effects since in the asymptotic approach all terms of $O(\varepsilon^2)$ are neglected.

It should be pointed out that the asymptotic curve in Fig. 7 follows from the general asymptotic results for this flow and heat transfer situation. It holds for all Newtonian fluids, while the non-asymptotic results from the study by Potter and Graber were calculated for water in the specific temperature range of their study.

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